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Comparison theorems for separable wavelet frames

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Abstract

It is known that wavelet frames do not exhibit a Nyquist density. Even so, this paper shows that the affine densities of the sets $U \times V$ and $S \times T$ affect the frame properties of $\{u^{-\frac{1}{2}} f(\frac{x}{u} - v)\}_{u\in U, v\in V}$ and $\{s^{-\frac{1}{2}} g(\frac{x}{s} - t)\}_{s\in S, t\in T}$. In particular, it is shown that there is a relationship between the densities of the dilation sets U and S and weighted admissibility constants of f and g. This relationship implies a comparison theorem, whereby the affine densities of $U \times V$ and $S \times T$ are proportional, with proportionality constant depending on the frame bounds and the admissibility constants of f and g. These results are also extended to wavelet frame sequences.

Keywords: Affine group; Comparison theorem; Density; Frames; Homogeneous Approximation Property; Nyquist density; Wavelets

1. Introduction

A frame for a separable Hilbert space H gives stable, but usually redundant, series representations of each element in the space. The best-known frames for function spaces are *coherent state frames* of the form $\{\sigma(x) f\}_{x\in X}$ where σ is a unitary representation of a locally compact group G on H and X is some collection of points in G. In particular, wavelet frames and Gabor frames for $L^2(\mathbb{R})$ have this form, as do Fourier frames for $L^2(I)$ where I is a compact interval.

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The density of X in G, which is in some sense the "average" number of points of X in a subset of G with unit measure, influences the properties of the frame. In the case that G is a locally compact abelian (LCA) group, much is known about the relationship between the frame properties of $\{\sigma(x)f\}_{x\in X}$ and the density of X. In particular, X must have density larger than some fixed "critical density" or Nyquist density in order for $\{\sigma(x)f\}_{x\in X}$ to be a frame. This critical Beurling density phenomenon underlies the classic Nyquist–Shannon Sampling Theorem and the work of Landau, both of which characterize frames of exponentials for $L^2(I)$ (see [13,15,12]). The Nyquist density properties of arbitrary Gabor frames were derived by Ramanthan and Steger in [14] (see [8] for an exposition of the history of density theorems for Gabor frames as well as extensive references). These critical density results were extended to arbitrary LCA groups in [2]. The Homogeneous Approximation Property (HAP), originally developed in [14], is a powerful tool for analyzing frames. As demonstrated in [2,7,9], it is the HAP for LCA frames that gives rise to the critical density that these frames obey. The HAP for LCA frames also gives rise to a "comparison theorem" as in Theorem 7 in [2]: if $\{\sigma(x)f\}_{x\in X}$ is a frame with bounds A, B and $\{\sigma(y)g\}_{y\in Y}$ is a frame with bounds E, F then

$$\frac{A\,\|g\|^2}{F\,\|f\|^2} \le \frac{D(X,p,c)}{D(Y,p,c)} \le \frac{B\,\|g\|^2}{E\,\|f\|^2}, \tag{1}$$

where $D(X,p,c)$ is some measure of the density of X, defined precisely in Section 2.

If σ is a unitary representation of a locally compact *non-abelian* group, then a frame $\{\sigma(x)f\}_{x\in X}$ need not demonstrate a critical density phenomenon. In particular, wavelet frames, which arise from the representation of the affine group on $L^2(\mathbb{R})$, are well-known for not having a critical density. For any $a>1$, $b\neq 0$ there is some ψ so that $\{a^{-\frac{m}{2}}\psi\left(\frac{x}{a^m}-bn\right)\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, which implies that for any positive number d, there is a wavelet frame for $L^2(\mathbb{R})$ with density d (see [4]). This fact still holds when we consider ψ having some fixed admissibility coefficient (see [5]), and in the case that $\{a^{-\frac{m}{2}}\psi\left(\frac{x}{a^m}-bn\right)\}_{m,n\in\mathbb{Z}}$ is a Riesz basis, $\{a^{-\frac{m}{2}}\psi\left(\frac{x}{a^m}-\beta n\right)\}_{m,n\in\mathbb{Z}}$ is still a Riesz basis for all β near b (see [1]). In light of these facts, it is surprising that wavelet frames do satisfy a homogeneous approximation property. In [10], the authors prove a HAP for wavelet frames, and for suitable wavelet frames $\{\sigma(x)f\}_{x\in X}$ and $\{\sigma(y)g\}_{y\in Y}$, the HAP gives one-sided density estimates: for each $\varepsilon>0$, there is some $R(g,\varepsilon)$ so that

$$\frac{1-\varepsilon}{\mathrm{e}^{R(g,\varepsilon)}} \le \frac{D(X,p,c)}{D(Y,p,c)}. \tag{2}$$

However, the HAP cannot imply a critical density or a two-sided estimate like (1). These results are generalized to arbitrary locally compact groups in [6], although the results are qualitative in nature, in contrast to the very precise results known for LCA frames.

In this paper, we will compare separable wavelet frames of the form $\{\sigma(u,v)f\}_{u\in U,v\in V}$ and $\{\sigma(s,t)g\}_{s\in S,t\in T}$. Since the best-known wavelet frames have this form, these results are applicable to a broad class of familiar wavelets as well as certain more general irregular wavelet systems. Our main result is a HAP for separable wavelet frames that is both more powerful that the usual HAP in some sense but less powerful in another. This HAP result allows us to delineate relationships between the densities of U, V, S and T, the admissibility constants of f, g and the frame bounds of the sequences $\{\sigma(u,v)f\}_{u\in U,v\in V}$ and $\{\sigma(s,t)g\}_{s\in S,t\in T}$. As a consequence, we obtain a comparison theorem for separable wavelet frames analogous to (1). Our comparison theorem is interesting because it shows a new similarity between wavelet frames and LCA frames. Both LCA frames and certain wavelet frames have a HAP and have a two-sided comparison theorem. Yet LCA frames have a critical density, while wavelet frames do not.

Separable wavelet frames allow us to independently analyze the translation and dilation parameters comprising the frame. Our main result concerns the dilation indices. For suitable $U, S \subset \mathbb{R}^+$ and suitable $f, g \in L^2(\mathbb{R})$ we show that

$$0 = \lim_{M\longrightarrow\infty} \frac{1}{2M} \left(\sum_{s\in S\cap a_M[\mathrm{e}^{-M},\mathrm{e}^M]} \sum_{u\in U} \int \left|\hat{g}(sw)\right|^2 |\hat{f}(uw)|^2 \frac{\mathrm{d}w}{|w|} \right.$$
$$\left. - \sum_{u\in U\cap a_M[\mathrm{e}^{-M},\mathrm{e}^M]} \sum_{s\in S} \int \left|\hat{g}(sw)\right|^2 |\hat{f}(uw)|^2 \frac{\mathrm{d}w}{|w|} \right)$$

for all sequences $\{a_M\}_{M\in\mathbb{N}} \subset \mathbb{R}^+$. For separable wavelet frames whose translations form a Fourier frame, this result is a type of HAP on $\mathbb{R}^+$ because it insures that functions are well-approximated by finitely many dilations and infinitely many translations. However, it is in fact more powerful than the usual HAP because it insures simultaneous approximation by $\{\sigma(u, v)f\}_{u\in U,v\in V}$ and $\{\sigma(s, t)g\}_{s\in S,t\in T}$.

As a consequence of our HAP, we obtain a comparison theorem for the densities of two wavelet frames. In particular, if $\{\sigma(u, v)f\}_{(u,v)\in U\times V}$, $\{\sigma(s, t)g\}_{(s,t)\in S\times T}$ are frames for $L^2(\mathbb{R})$ with frame bounds A, B and E, F, respectively then

$$\frac{A\,C_g}{F\,C_f} \le \frac{D(U \times V, c, p)}{D(S \times T, c, p)} \le \frac{B\,C_g}{E\,C_f}$$

for all suitable $f, g \in L^2(\mathbb{R})$, $U, S \subset \mathbb{R}^+$ and $V, T \subset \mathbb{R}$, where C_f, C_g are the admissibility constants of f, g.

The paper is organized into five sections: Section 2 contains background information and preliminary lemmas; Section 3 contains the main result and its proof; The applications of the main result to wavelet frames are explored in Section 4; These results are extended to certain wavelet frame sequences in Section 5.

2. Preliminaries

2.1. *Affine group*

The affine group $\mathbb{A}$ is the set $\mathbb{R}^+ \times \mathbb{R}$ with multiplication

$$(a, b)(x, y) = \left(ax, y + \frac{b}{x}\right).$$

For $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$, we let $\sigma(a, b)$ denote the $L^2(\mathbb{R})$ operator $D_a T_b$, where D_a denotes the dilation $D_a f(t) = a^{-\frac{1}{2}} f(\frac{t}{a})$ and T_b denotes the translation $T_b f(t) = f(t - b)$. Thus σ is a unitary representation of the affine group on $L^2(\mathbb{R})$. Let μ denote the left Haar measure of the affine group on $L^2(\mathbb{R})$; that is, $\mathrm{d}\mu(a, b) = \frac{\mathrm{d}a}{a}\,\mathrm{d}b$.

2.2. *Continuous wavelet transform*

The continuous wavelet transform of $h \in L^2(\mathbb{R})$ with respect to $f \in L^2(\mathbb{R})$ is

$$W_f h(a, b) = \langle h, \sigma(a, b)f\rangle, \quad (a, b) \in \mathbb{A}.$$

A function $f \in L^2(\mathbb{R})$ is *admissible* if

$$C_f = \int_{\mathbb{R}} \left|\hat{f}(w)\right|^2 \frac{\mathrm{d}w}{|w|} < \infty.$$

If f is admissible, then C_f is called the *admissibility constant* of f. In this case, the inversion formula

$$h(t) = C_f^{-1} \int_{\mathbb{A}} W_f h(a,b)\,\sigma(a,b) f(t)\,\mathrm{d}\mu(a,b)$$

holds weakly for all $h \in L^2(\mathbb{R})$.

2.3. Wavelet frames and Bessel sequences

A *wavelet frame* for $L^2(\mathbb{R})$ with frame bounds A, B is a sequence $\{\sigma(x) f\}_{x\in X}$, where $f \in L^2(\mathbb{R})$ and $X \subset \mathbb{A}$, satisfying

$$\forall h \in L^2(\mathbb{R}), \quad A\,\|h\|^2 \le \sum_{x\in X} \left|W_f h(x)\right|^2 \le B\,\|h\|^2.$$

If $\{\sigma(x) f\}_{x\in X}$ is a frame for $L^2(\mathbb{R})$, there is a dual sequence $\{\tilde{f}_x\}_{x\in X} \subset L^2(\mathbb{R})$ such that

$$h = \sum_{x\in X} W_f h(x)\,\tilde{f}_x = \sum_{x\in X} \langle h, \sigma(x) f\rangle\,\tilde{f}_x = \sum_{x\in X} \langle h, \tilde{f}_x\rangle \sigma(x) f$$

for all $h \in L^2(\mathbb{R})$, and the sequence $\{\tilde{f}_x\}_{x\in X}$ can be chosen to be a frame for $L^2(\mathbb{R})$.

A sequence $\{\sigma(x) f\}_{x\in X}$ satisfying

$$\forall h \in \overline{\mathrm{span}}\{\sigma(x) f\}_{x\in X}, \quad A\,\|h\|^2 \le \sum_{x\in X} \left|W_f h(x)\right|^2 \le B\,\|h\|^2$$

is called a *wavelet frame sequence*. A sequence $\{\sigma(x) f\}_{x\in X}$ satisfying

$$\forall h \in L^2(\mathbb{R}), \quad \sum_{x\in X} \left|W_f h(x)\right|^2 \le B\,\|h\|^2$$

is a *Bessel sequence* with Bessel bound B.

2.4. Density

2.4.1. Free ultrafilters

Intuitively, the density of X in G should be the "average" number of points of X in a subset of G with unit measure. To make this idea precise, we use free ultrafilters.

Definition 2.1. A collection p of subsets of $\mathbb{N}$ is a *filter* if

(a) $\emptyset \notin p$,
(b) if $A, B \in p$ then $A \cap B \in p$, and
(c) if $A \in p$ and $A \subset B \subset \mathbb{N}$ then $B \in p$.

A filter p is an *ultrafilter* if it is maximal, i.e. if p' is a filter and $p \subset p'$ then $p' = p$.
An ultrafilter p is a *free ultrafilter* if p contains no finite sets.

Definition 2.2. Suppose p is an ultrafilter and $\{c_n\}_{n\in\mathbb{N}}$ a sequence in $\mathbb{C}$. We say $\{c_n\}_{n\in\mathbb{N}}$ *converges* to $c \in \mathbb{C}$ *with respect to* p if for every $\epsilon > 0$ there exists $A \in p$ with $|c_n - c| < \varepsilon$ for all $n \in A$. In this case we write $p\text{-}\lim c_n = c$.

The basic convergence properties of free ultrafilters are summarized in the following proposition.

Proposition 2.3. *Assume p is a free ultrafilter.*

(a) *p-limits are unique.*
(b) *If $\{c_n\}_{n\in\mathbb{N}}$ is a bounded sequence of complex scalars then $p\text{-}\lim c_n$ exists and is an accumulation point of $\{c_n\}_{n\in\mathbb{N}}$.*
(c) *If c is an accumulation point of $\{c_n\}_{n\in\mathbb{N}}$, there is an ultrafilter p with $p\text{-}\lim c_n = c$. In particular $\liminf c_n$ and $\limsup c_n$ are p-limits.*
(d) *If $\lim_{n\to\infty} c_n = c$ then $p\text{-}\lim c_n = c$.*
(e) *p-limits are linear.*
(f) *p-limits respect products.*

2.4.2. General density

Definition 2.4. Let G be a locally compact group with left Haar measure μ, and let $\{Q_M\}_{M\in\mathbb{N}} \subset G$ be a sequence of compact sets satisfying $Q_M \subset Q_{M+1}$ for all $M \in \mathbb{N}$ and $\cup Q_M = G$. Let X be any collection of points in G. For any free ultrafilter p and each sequence $c = \{c_M\}_{M\in\mathbb{N}} \subset G$, we define the *density* of X with respect to p and c to be

$$D_G(X, p, c) = p\text{-}\lim \frac{|X \cap c_M Q_M|}{\mu(Q_M)}.$$

The *upper density* of X is

$$D_G^+(X) = \limsup_{M\to\infty} \sup_{g\in G} \frac{|X \cap gQ_M|}{\mu(Q_M)}$$

while the *lower density* of X is

$$D_G^-(X) = \liminf_{M\to\infty} \inf_{g\in G} \frac{|X \cap gQ_M|}{\mu(Q_M)},$$

where $c_M Q_M$, gQ_M denote left multiplication by by c_M, g, respectively.

For each free ultrafilter p and each sequence $c = \{c_M\}_{M\in\mathbb{N}} \subset G$, we have

$$D_G^-(X) \le D_G(X, p, c) \le D_G^+(X).$$

Furthermore, there are p, c so that $D_G^+(X) = D(X, p, c)$. Similarly there exist p, c so that $D_G^-(X) = D(X, p, c)$.

In general, if there are p, c so that $D_G(X, p, c) = \infty$ then no $\{\sigma(x) f\}_{x\in X}$ will be a frame. To avoid such sets we make the following definition.

Definition 2.5. Suppose G is a locally compact group and X is a collection of points in G. If for any compact $U \subset G$, there is some finite K so that

$$\left\| \sum_{x\in X} \chi_{xU} \right\|_\infty \le K$$

then X is *relatively separated*.

2.4.3. Affine density

We will consider affine density with respect to the choice of sets $\{Q_M\}_{M\in\mathbb{N}}$ given by $Q_M = [\mathrm{e}^{-M}, \mathrm{e}^{M}] \times [-M, M]$. Henceforth $D_{\mathbb{A}}(X, p, c)$, $D^+_{\mathbb{A}}(X)$ and $D^-_{\mathbb{A}}(X)$ are defined as in Definition 2.4 with respect to this particular choice of Q_M. The set Q_M is a rectangle in $\mathbb{A}$ centered at $(1, 0)$, and $\mu(Q_M) = 4M^2$.

The following lemma ensures that relatively separated sets in the affine group have finite density (see Lemma 3.1 in [16] for proof).

Lemma 2.6. *If X is a relatively separated set in $\mathbb{A}$, then there is some finite K so that*

$$D_{\mathbb{A}}(X, p, c) \le K$$

for all free ultrafilters p and all sequences $c = \{c_M\}_{M\in\mathbb{N}} \subset \mathbb{A}$. In particular, $D^+_{\mathbb{A}}(X) < \infty$.

2.4.4. Density of $\mathbb{R}^+$, $\mathbb{R}$

In addition to density of sets in $\mathbb{A}$, it will be useful to measure the densities of subsets of $\mathbb{R}^+$ and $\mathbb{R}$. We fix $I_M = [\mathrm{e}^{-M}, \mathrm{e}^{M}]$. Following Definition 2.4, for $U \subset \mathbb{R}^+$ and $a = \{a_M\} \subset \mathbb{R}^+$ we set

$$D_{\mathbb{R}^+}(U, p, a) = p\text{-lim}\, \frac{|U \cap [a_M \mathrm{e}^{-M}, a_M \mathrm{e}^{M}]|}{2M} = p\text{-lim}\, \frac{|U \cap a_M I_M|}{2M}.$$

For $V \subset \mathbb{R}$ and $b = \{b_M\} \subset \mathbb{R}$ we set

$$D_{\mathbb{R}}(V, p, b) = p\text{-lim}\, \frac{|V \cap (b_M + [-M, M])|}{2M}.$$

The following lemma relates density in $\mathbb{A}$ to density in $\mathbb{R}^+$ and $\mathbb{R}$.

Lemma 2.7. *Suppose $U \subset \mathbb{R}^+$ and $V \subset \mathbb{R}$. For any sequence $\{c_M\} = \{(a_M, b_M)\} \subset \mathbb{A}$ and any free ultrafilter p we have*

$$D^-_{\mathbb{R}}(V)\, D_{\mathbb{R}^+}(U, a, p) \le D_{\mathbb{A}}(U \times V, c, p) \le D^+_{\mathbb{R}}(V)\, D_{\mathbb{R}^+}(U, a, p).$$

Proof. Notice that $(u, v) \in c_M Q_M$ if and only if $u = a_M x$ for some $x \in [\mathrm{e}^{-M}, \mathrm{e}^{M}]$ and $v = y + \frac{a_M b_M}{u}$ for some $y \in [-M, M]$. Thus

$$\begin{aligned} p\text{-lim}\, \frac{|U \times V \cap c_M Q_M|}{\mu(Q_M)} &\le \left(p\text{-lim}\, \frac{|U \cap a_M[\mathrm{e}^{-M}, \mathrm{e}^{M}]|}{2M} \right) \\ &\quad \times \left(p\text{-lim} \sup_{u \in U \cap a_M[\mathrm{e}^{-M}, \mathrm{e}^{M}]} \frac{|V \cap \frac{a_M b_M}{u} + [-M, M]|}{2M} \right) \\ &\le D^+_{\mathbb{R}}(V) \cdot p\text{-lim}\, \frac{|U \cap a_M[\mathrm{e}^{-M}, \mathrm{e}^{M}]|}{2M} \end{aligned}$$

and similarly

$$p\text{-lim}\, \frac{|U \times V \cap c_M Q_M|}{\mu(Q_M)} \ge D^-_{\mathbb{R}}(V) \cdot p\text{-lim}\, \frac{|U \cap a_M[\mathrm{e}^{-M}, \mathrm{e}^{M}]|}{2M}. \quad \square$$

2.5. Fourier frames and separable wavelet frames

Definition 2.8. We say that $\mathcal{E}(T) = \left\{e^{2\pi i t x}\right\}_{t\in T}$ is a *Fourier frame* if there is some r so that $\mathcal{E}(T)$ is a frame for $L^2[-r,r]$.

If $\mathcal{E}(T)$ is a Fourier frame then the frame and Bessel sequence properties of a sequence of the form $\{\sigma(s,t)g\}_{(s,t)\in S\times T}$ are largely determined by the behavior of the function $\sum_{s\in S}|\hat{g}(sx)|^2$. For this reason, we make the following definition.

Definition 2.9. Let $S\subset\mathbb{R}^+$. We say that g is *Chui–Shi bounded with respect to S* if there is some finite K such that

$$\sum_{s\in S}\left|\hat{g}(sx)\right|^2\le K\quad \text{a.e.}$$

It was proved in [3] that for regular wavelet frames $\{\sigma(a^m,bn)g\}$, the function $\sum_m\left|\hat{g}(a^m x)\right|^2$ is bounded almost everywhere. This was generalized in [18] to separable wavelet frames whose translations give rise to a Fourier frame.

Definition 2.10. Given a free ultrafilter p, sequence $c=\{c_M\}_{M\in\mathbb{N}}=\{(a_M,b_M)\}_{M\in\mathbb{N}}\subset\mathbb{A}$ and admissible f,g generating wavelet Bessel sequences $\mathcal{G}=\{\sigma(s,t)g\}_{(s,t)\in S\times T}$ and $\mathcal{F}=\{\sigma(u,v)f\}_{(u,v)\in U\times V}$, we define the *relative admissibility measure* of $\mathcal{F}$ with respect to $\mathcal{G}$ to be

$$\mu_{\mathcal{F},\mathcal{G}}(p,c)=p\text{-}\lim\frac{1}{|U\cap a_M I_M|}\sum_{u\in U\cap a_M I_M}\sum_{s\in S}\int\left|\hat{g}(sx)\right|^2|\hat{f}(ux)|^2\frac{dx}{|x|}.$$

If g is Chui–Shi bounded with respect to S, then $\mu_{\mathcal{F},\mathcal{G}}(p,c)$ is a type of average admissibility constant for f.

2.6. Localization

In this subsection we develop results that allow us to estimate sums of the form $\sum_{u\in U\cap a_M I_M^c}\sum_{s\in S}\int\left|\hat{g}(sx)\right|^2|\hat{f}(ux)|^2\frac{dx}{|x|}$.

Definition 2.11. Suppose $f,g\in L^2(\mathbb{R})$. We say that f,g are a *localized pair* if

$$\int_{[0,\infty)}\left(\sup_{c\in[ye^{-1},ye]}\int|\hat{g}(x)|^2|\hat{f}(cx)|^2\frac{dx}{|x|}\right)\frac{dy}{y}<\infty.$$

Notice that

$$\int_{[0,\infty)}\sup_{c\in[ye^{-1},ye]}\int|\hat{g}(x)|^2|\hat{f}(cx)|^2\frac{dx}{|x|}\frac{dy}{y}$$
$$=\int_{[0,\infty)}\sup_{c\in[ye^{-1},ye]}\int|\hat{g}(cx)|^2|\hat{f}(x)|^2\frac{dx}{|x|}\frac{dy}{y}$$

so that localization is a symmetric relation.

The following lemma gives a class of functions that form a localized pair with any admissible wavelet. A generalization of this proof technique shows that any function in $L^2\cap L^\infty$ whose

Fourier transform is supported in $[-\Omega_1, -\Omega_0] \cup [\Omega_0, \Omega_1]$ forms a localized pair with any admissible wavelet.

Lemma 2.12. *Fix $a > 1$. Every admissible function f forms a localized pair with the function g whose Fourier transform is $\hat{g} = \chi_{[-1,-a^{-1}]\cup[a^{-1},1]}$.*

Proof. Fix an admissible function f. We have $\hat{g} = \chi_{[-1,-a^{-1}]\cup[a^{-1},1]}$. Then

$$\int_{[0,\infty)} \left|\hat{g}(x)\right|^2 |\hat{f}(cx)|^2 \frac{\mathrm{d}x}{|x|} = \int_{[ca^{-1},c]} |\hat{f}(x)|^2 \frac{\mathrm{d}x}{|x|}.$$

For $\mathrm{e}^m \le y \le \mathrm{e}^{m+1}$ and $c \in [y\mathrm{e}^{-1}, y\mathrm{e}]$ we have $[ca^{-1}, c] \subset [a^{-1}\mathrm{e}^{m-1}, \mathrm{e}^{m+2}]$. Choose $k > 0$ so that $a \le \mathrm{e}^k$. Then $[a^{-1}\mathrm{e}^{m-1}, \mathrm{e}^{m+2}] \subset [a^{-1}\mathrm{e}^{m-1}, a^{-1}\mathrm{e}^{m+k+2}]$. We have

$$\begin{aligned}
&\int_{[0,\infty)} \sup_{c\in[y\mathrm{e}^{-1},y\mathrm{e}]} \int_{[0,\infty)} \left|\hat{g}(x)\right|^2 |\hat{f}(cx)|^2 \frac{\mathrm{d}x}{|x|}\frac{\mathrm{d}y}{y} \\
&= \sum_{m\in\mathbb{Z}} \int_{[\mathrm{e}^m,\mathrm{e}^{m+1})} \sup_{c\in[y\mathrm{e}^{-1},y\mathrm{e}]} \int_{[ca^{-1},c]} |\hat{f}(x)|^2 \frac{\mathrm{d}x}{|x|}\frac{\mathrm{d}y}{y} \\
&\le \sum_{m\in\mathbb{Z}} \int_{[\mathrm{e}^m,\mathrm{e}^{m+1})} \int_{[a^{-1}\mathrm{e}^{m-1},\mathrm{e}^{m+2}]} |\hat{f}(x)|^2 \frac{\mathrm{d}x}{|x|}\frac{\mathrm{d}y}{y} \\
&= \sum_{m\in\mathbb{Z}} \int_{[a^{-1}\mathrm{e}^{m-1},\mathrm{e}^{m+2}]} |\hat{f}(x)|^2 \frac{\mathrm{d}x}{|x|} \\
&\le \sum_{m\in\mathbb{Z}} \int_{[a^{-1}\mathrm{e}^{m-1},a^{-1}\mathrm{e}^{m+k+2}]} |\hat{f}(x)|^2 \frac{\mathrm{d}x}{|x|} \\
&\le (k+3)C_f.
\end{aligned}$$

Similar estimates hold for

$$\int_{(-\infty,0]} \left|\hat{g}(x)\right|^2 |\hat{f}(cx)|^2 \frac{\mathrm{d}x}{|x|}. \quad \square$$

The following result is a special case of Lemma 1 in [6].

Lemma 2.13. *Suppose that $R \subset \mathbb{R}^+$ is relatively separated. If f, g are a localized pair then there is some finite K so that*

$$\sum_{r\in R\cap I_M^c} \int |\hat{g}(x)|^2 |\hat{f}(rx)|^2 \frac{\mathrm{d}x}{|x|} \le K \int_{I_{M-1}^c} \sup_{r\in[y\mathrm{e}^{-1},y\mathrm{e}]} \int \left|\hat{g}(x)\right|^2 |\hat{f}(rx)|^2 \frac{\mathrm{d}x}{|x|}\frac{\mathrm{d}y}{y}$$

for all $M > 1$.

3. The main result

We begin by showing that for suitable f, g, certain average admissibility constants of f, g are proportional. We need not have wavelet frames to derive this result.

Theorem 3.1. *Suppose that U and S are relatively separated in $\mathbb{R}^+$ and $f, g \in L^2(\mathbb{R})$ are admissible, form a localized pair, and are Chui–Shi bounded with respect to U, S, respectively.*

Then for any sequence $\{a_M\} \subset \mathbb{R}^+$, *we have*

$$0 = \lim_{M\to\infty} \frac{1}{2M} \left(\sum_{s\in S\cap a_M I_M} \sum_{u\in U} \int \left|\hat{g}(sw)\right|^2 |\hat{f}(uw)|^2 \frac{\mathrm{d}w}{|w|} - \sum_{u\in U\cap a_M I_M} \sum_{s\in S} \int \left|\hat{g}(sw)\right|^2 |\hat{f}(uw)|^2 \frac{\mathrm{d}w}{|w|} \right).$$

Proof. Fix $\varepsilon > 0$. Since f, g are a localized pair, we can choose $M_\varepsilon \in \mathbb{N}$ so that

$$\int_{I^c_{M_\varepsilon - 1}} \sup_{r\in[y\mathrm{e}^{-1}, y\mathrm{e}]} \int \left|\hat{g}(x)\right|^2 |\hat{f}(rx)|^2 \frac{\mathrm{d}x}{|x|} \frac{\mathrm{d}y}{y} < \varepsilon$$

and

$$\int_{I^c_{M_\varepsilon - 1}} \sup_{r\in[y\mathrm{e}^{-1}, y\mathrm{e}]} \int \left|\hat{g}(rx)\right|^2 |\hat{f}(x)|^2 \frac{\mathrm{d}x}{|x|} \frac{\mathrm{d}y}{y} < \varepsilon.$$

By Lemma 2.13, we can choose $K_1 < \infty$ so that for all $M > 1$ we have

$$\sum_{u\in U\cap I^c_M} \int \left|\hat{g}(x)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} \le K_1 \int_{I^c_{M-1}} \sup_{r\in[y\mathrm{e}^{-1}, y\mathrm{e}]} \int \left|\hat{g}(x)\right|^2 |\hat{f}(rx)|^2 \frac{\mathrm{d}x}{|x|} \frac{\mathrm{d}y}{y}$$

and

$$\sum_{s\in S\cap I^c_M} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(x)|^2 \frac{\mathrm{d}x}{|x|} \le K_1 \int_{I^c_{M-1}} \sup_{r\in[y\mathrm{e}^{-1}, y\mathrm{e}]} \int \left|\hat{g}(rx)\right|^2 |\hat{f}(x)|^2 \frac{\mathrm{d}x}{|x|} \frac{\mathrm{d}y}{y}.$$

Since f, g are Chui–Shi bounded with respect to U, S, we can choose $K_2 < \infty$ so that

$$\sum_{s\in S} \left|\hat{g}(sw)\right|^2 < K_2 \quad \text{a.e. and} \quad \sum_{u\in U} \left|\hat{f}(uw)\right|^2 < K_2 \quad \text{a.e.}$$

Since U and S are relatively separated in $\mathbb{R}^+$, we can choose $K_3 < \infty$ so that for all $M > 0$ and $r \in \mathbb{R}^+$ we have

$$\left|S \cap r[\mathrm{e}^{-M}, \mathrm{e}^M]\right| = |S \cap r I_M| \le 2K_3 M$$

and

$$\left|U \cap r[\mathrm{e}^{-M}, \mathrm{e}^M]\right| = |U \cap r I_M| \le 2K_3 M.$$

Write

$$\begin{aligned}
&\sum_{s\in S\cap a_M I_M} \sum_{u\in U} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} - \sum_{u\in U\cap a_M I_M} \sum_{s\in S} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} \\
&\quad = \sum_{s\in S\cap a_M I_M} \sum_{u\in U\cap a_M I^c_M} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} \\
&\qquad - \sum_{u\in U\cap a_M I_M} \sum_{s\in S\cap a_M I^c_M} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{s\in S\cap a_M I_M} \sum_{u\in U\cap a_M I^c_{M+M_\varepsilon}} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} \\
&\quad + \sum_{s\in S\cap a_M I_M} \sum_{u\in U\cap a_M (I_{M+M_\varepsilon}\setminus I_M)} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} \\
&\quad - \sum_{u\in U\cap a_M I_M} \sum_{s\in S\cap a_M I^c_{M+M_\varepsilon}} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} \\
&\quad - \sum_{u\in U\cap a_M I_M} \sum_{s\in S\cap a_M (I_{M+M_\varepsilon}\setminus I_M)} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} \\
&= T_1 + T_2 - T_3 - T_4.
\end{aligned}
$$

We can estimate T_1 by noting that for $s \in S\cap a_M I_M$ and $u \in U\cap a_M I^c_{M+M_\varepsilon}$, we have $\frac{s}{u} \in I^c_{M_\varepsilon}$. Thus

$$
\begin{aligned}
|T_1| &= \sum_{s\in S\cap a_M I_M} \sum_{u\in U\cap a_M I^c_{M+M_\varepsilon}} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} \\
&\le |S\cap a_M I_M| \sup_{s\in S\cap a_M I_M} \sum_{u\in U\cap a_M I^c_{M+M_\varepsilon}} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} \\
&\le |S\cap a_M I_M|\, K_1 \int_{I^c_{M_\varepsilon - 1}} \sup_{r\in[y\mathrm{e}^{-1},\, y e]} \int \left|\hat{g}(x)\right|^2 |\hat{f}(rx)|^2 \frac{\mathrm{d}x}{|x|}\,\frac{\mathrm{d}y}{y} \\
&\le 2MK_1K_3\varepsilon.
\end{aligned}
$$

We estimate T_2 by

$$
\begin{aligned}
|T_2| &= \sum_{s\in S\cap a_M I_M} \sum_{u\in U\cap a_M (I_{M+M_\varepsilon}\setminus I_M)} \int \left|\hat{g}(sx)\right|^2 |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} \\
&\le \sum_{u\in U\cap a_M (I_{M+M_\varepsilon}\setminus I_M)} K_2 \int |\hat{f}(ux)|^2 \frac{\mathrm{d}x}{|x|} \\
&= \sum_{u\in U\cap a_M (I_{M+M_\varepsilon}\setminus I_M)} K_2 C_f \\
&\le K_2K_3C_f 2M_\varepsilon.
\end{aligned}
$$

Similarly, we can show

$$|T_3| \le 2MK_1K_3\varepsilon$$

and

$$|T_4| \le 2K_2K_3C_gM_\varepsilon.$$

Thus

$$
\begin{aligned}
&\frac{1}{2M}\left|\sum_{s\in S\cap a_M I_M}\sum_{u\in U}\int \left|\hat{g}(sw)\right|^2 |\hat{f}(uw)|^2 \frac{\mathrm{d}w}{|w|} - \sum_{u\in U\cap a_M I_M}\sum_{s\in S}\int \left|\hat{g}(sw)\right|^2 |\hat{f}(uw)|^2 \frac{\mathrm{d}w}{|w|}\right| \\
&\quad\le \frac{|T_1| + |T_2| + |T_3| + |T_4|}{2M}
\end{aligned}
$$

$$\leq 4K_1K_3\varepsilon + 2K_2K_3(C_g + C_f)\frac{M_\varepsilon}{M}$$
$$\to 4K_1K_3\varepsilon \quad \text{as } M \to \infty.$$

Since ε is arbitrary, the result follows. □

For separable wavelet frames and frame sequences in $L^2(\mathbb{R})$, Theorem 3.1 can be restated as a useful relationship between the relative admissibility measure of a frame and the density of its dilation parameters.

Corollary 3.2. *Suppose U, S are relatively separated in $\mathbb{R}^+$ and $f, g \in L^2(\mathbb{R})$ are admissible, form a localized pair, and are Chui–Shi bounded with respect to U, S, respectively. Let $\mathcal{G} = \{\sigma(s,t)g\}_{(s,t)\in S\times T}$ and $\mathcal{F} = \{\sigma(u,v)f\}_{(u,v)\in U\times V}$. Then for any sequence $a = \{a_M\} \subset \mathbb{R}^+$,*

$$\mu_{\mathcal{F},\mathcal{G}}(p,c) \cdot D_{\mathbb{R}^+}(U,p,a) = \mu_{\mathcal{G},\mathcal{F}}(p,c) \cdot D_{\mathbb{R}^+}(S,p,a),$$

where $c = \{(a_M, b_M)\} \subset \mathbb{A}$ for any sequence $\{b_M\} \subset \mathbb{R}$.

4. Wavelet frames for $L^2(\mathbb{R})$

In this section, we apply Theorem 3.1 and Corollary 3.2 to wavelet frames for $L^2(\mathbb{R})$ to derive a comparison theorem.

Theorem 4.1. *Suppose the following conditions hold.*

(a) *$f, g \in L^2(\mathbb{R})$ are admissible and form a localized pair.*
(b) *U and S are relatively separated in $\mathbb{R}^+$.*
(c) *$V \subset \mathbb{R}$ and $\mathcal{E}(V) = \left\{e^{2\pi i v w}\right\}_{v\in V}$ is a frame for $L^2[-r_V, r_V]$ with bounds A_V, B_V.*
(d) *$T \subset \mathbb{R}$ and $\mathcal{E}(T) = \left\{e^{2\pi i t w}\right\}_{t\in T}$ is a frame for $L^2[-r_T, r_T]$ with frame bounds E_T, F_T.*
(e) *$\mathcal{F} = \{\sigma(u,v)f\}_{(u,v)\in U\times V}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B.*
(f) *$\mathcal{G} = \{\sigma(s,t)g\}_{(s,t)\in S\times T}$ is a frame for $L^2(\mathbb{R})$ with frame bounds E, F.*

Then for any free ultrafilter p and any sequence $a = \{a_M\} \subset \mathbb{R}^+$

$$\frac{A\, E_T\, C_g}{F\, B_V\, C_f} \leq \frac{D_{\mathbb{R}^+}(U,a,p)}{D_{\mathbb{R}^+}(S,a,p)} \leq \frac{B\, F_T\, C_g}{E\, A_V\, C_f}.$$

Proof. By the main theorem in [18], we have

$$\frac{A}{B_V} \leq \sum_{u\in U} |\hat{f}(uw)|^2 \leq \frac{B}{A_V} \quad \text{a.a. } w \in \mathbb{R}.$$

Hence

$$\frac{A}{B_V} C_g \leq \sum_{u\in U} \int \left|\hat{g}(sw)\right|^2 |\hat{f}(uw)|^2 \frac{\mathrm{d}w}{|w|} \leq \frac{B}{A_V} C_g,$$

which implies

$$\frac{A}{B_V} C_g \leq \mu_{\mathcal{G},\mathcal{F}}(p,c) \leq \frac{B}{A_V} C_g$$

for all p and $c = \{(a_M, b_M)\} \subset \mathbb{A}$. Similarly,

$$\frac{E}{F_T} C_f \leq \mu_{\mathcal{F},\mathcal{G}}(p,c) \leq \frac{F}{E_T} C_f$$

for all p and $c = \{(a_M, b_M)\} \subset \mathbb{A}$. By Corollary 3.2 we have

$$\frac{D_{\mathbb{R}^+}(U, a, p)}{D_{\mathbb{R}^+}(S, a, p)} = \frac{\mu_{\mathcal{G},\mathcal{F}}(p, c)}{\mu_{\mathcal{F},\mathcal{G}}(p, c)}.$$

Combining these estimates proves the theorem. □

Although wavelet frames of the form $\mathcal{F} = \{\sigma(u, v) f\}_{(u,v)\in U\times V}$ with $\mathcal{E}(V) = \left\{e^{2\pi i v w}\right\}_{v\in V}$ a frame for $L^2[-r_V, r_V]$ constitute a broad class of separable wavelet frames, it is not true that every separable wavelet frame has this form. In Example 2.1 of [17], the authors construct a separable wavelet frame whose translations do not form a Fourier frame.

Corollary 4.2. *If $\mathcal{F} = \{\sigma(u, v) f\}_{(u,v)\in U\times V}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B and $\mathcal{E}(V) = \left\{e^{2\pi i v w}\right\}_{v\in V}$ a frame for $L^2[-r_V, r_V]$ with bounds A_V, B_V, then for any free ultrafilter p and any sequence $a = \{a_M\} \subset \mathbb{R}^+$ we have*

$$\frac{A}{B_V\, C_f} \le D_{\mathbb{R}^+}(U, a, p) \le \frac{B}{A_V\, C_f}.$$

Proof. Fix $r > 1$. Letting $g = \chi_{[-1,-r^{-1}]\cup[r^{-1},1]}$ and $\mathcal{G} = \{\sigma(r^m, n) g\}_{m,n\in\mathbb{Z}}$ we obtain an orthonormal basis for $L^2(\mathbb{R})$ with $C_g = \ln r$. Notice $\left\{e^{2\pi i n x}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$. We have

$$D_{\mathbb{R}^+}\left(\left\{r^m\right\}_{m\in\mathbb{Z}}, a, p\right) = \frac{1}{\ln r}$$

for all p, a. Since Lemma 2.12 ensures that f, g are a localized pair, the result follows from Theorem 4.1. □

We can use Theorem 4.1 to draw conclusions about the affine density of wavelet frames.

Theorem 4.3. *Suppose that the hypotheses of Theorem 4.1 hold. Then for any sequence $c = \{c_M\} \subset \mathbb{A}$ and free ultrafilter p we have*

$$\frac{A\, A_V\, E_T\, C_g}{F\, B_V\, F_T\, C_f} \le \frac{D_{\mathbb{A}}(U \times V, c, p)}{D_{\mathbb{A}}(S \times T, c, p)} \le \frac{B\, B_V\, F_T\, C_g}{E\, A_V\, E_T\, C_f}.$$

Proof. Write $\{c_M\} = \{(a_M, b_M)\}$. By Theorem 4.1, we have

$$\frac{A\, E_T\, C_g}{F\, B_V\, C_f} \le \frac{D_{\mathbb{R}^+}(U, a, p)}{D_{\mathbb{R}^+}(S, a, p)} \le \frac{B\, F_T\, C_g}{E\, A_V\, C_f},$$

where $a = \{a_M\}$. By Corollary 6 in [2], we obtain

$$A_V \le D^-_{\mathbb{R}}(V) \le D^+_{\mathbb{R}}(V) \le B_V$$

and

$$E_T \le D^-_{\mathbb{R}}(T) \le D^+_{\mathbb{R}}(T) \le F_T.$$

Combining these estimates with Lemma 2.7 proves the result. □

We recover the main theorem in [11] as a corollary to Theorem 4.3.

Corollary 4.4. *If $\mathcal{F} = \{\sigma(u,v)f\}_{(u,v)\in U\times V}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B and $\mathcal{E}(V) = \{e^{2\pi i v w}\}_{v\in V}$ is a frame for $L^2[-r_V, r_V]$ with bounds A_V, B_V, then for any free ultrafilter p and any sequence $c = \{c_M\} \subset \mathbb{A}$ we have*

$$\frac{A\,A_V}{B_V\,C_f} \le D_{\mathbb{A}}(U\times V, c, p) \le \frac{B\,B_V}{A_V\,C_f}.$$

Proof. This follows from Corollary 4.2 and Theorem 4.3. □

5. Wavelet frame sequences

It may appear that the crux of the proofs of Theorems 4.1 and 4.3 is the the estimates

$$\frac{A}{B_V} \le \sum_{u\in U} \left|\hat{f}(uw)\right|^2 \le \frac{B}{A_V} \quad \text{a.a. } w\in\mathbb{R} \tag{3}$$

and

$$\frac{E}{F_T} \le \sum_{s\in S} \left|\hat{g}(sw)\right|^2 \le \frac{F}{E_T} \quad \text{a.a. } w\in\mathbb{R}, \tag{4}$$

which are guaranteed by [18] when $\mathcal{F}, \mathcal{G}$ are frames for $L^2(\mathbb{R})$. However, this is not true. We can adapt our above approach to obtain similar comparison results for certain separable wavelet frame sequences for which the inequalities (3) and (4) need not hold.

Define an operator Δ by

$$(\Delta h)^\wedge(w) = \frac{\hat{h}(w)}{\sqrt{|w|}}.$$

A function h is admissible if and only if $h\in L^2(\mathbb{R})$ and $\Delta h \in L^2(\mathbb{R})$. The admissibility constant of h is $C_h = \|\Delta h\|_2^2$.

Theorem 5.1. *Suppose the following conditions hold.*

(a) *U and S are relatively separated in $\mathbb{R}^+$.*
(b) *$f, g \in L^2(\mathbb{R})$ are admissible, form a localized pair, have compact support, and are Chui–Shi bounded with respect to U, S, respectively.*
(c) *$V\subset\mathbb{R}$ and $\mathcal{E}(V) = \{e^{2\pi i v w}\}_{v\in V}$ is a frame for $L^2(\operatorname{supp}\hat{f})$ with bounds A_V, B_V.*
(d) *$T\subset\mathbb{R}$ and $\mathcal{E}(T) = \{e^{2\pi i t w}\}_{t\in T}$ is a frame for $L^2(\operatorname{supp}\hat{g})$ with frame bounds E_T, F_T.*
(e) *$S_{\mathcal{F}}$ is the frame operator for the sequence $\mathcal{F} = \{\sigma(u,v)f\}_{(u,v)\in U\times V}$.*
(f) *$S_{\mathcal{G}}$ is the frame operator for the sequence $\mathcal{G} = \{\sigma(s,t)g\}_{(s,t)\in S\times T}$.*

Then there exist constants $\alpha_{s,t}\in[B_V^{-1}, A_V^{-1}]$ and $\lambda_{u,v}\in[F_T^{-1}, E_T^{-1}]$ so that

$$0 = \lim_{M\to\infty}\frac{1}{2M}\left(\sum_{s\in S\cap a_M I_M}\alpha_{s,t}\,\langle\sigma(s,t)\Delta g,\ S_{\mathcal{F}}\sigma(s,t)\Delta g\rangle \right.$$
$$\left. - \sum_{u\in U\cap a_M I_M}\lambda_{u,v}\left\langle\sigma(u,v)\Delta f,\ S_{\mathcal{G}}\sigma(u,v)\Delta f\right\rangle\right)$$

for any sequence $a = \{a_M\}\subset\mathbb{R}^+$.

Proof. Note that

$$\begin{aligned}\langle\sigma(s,t)\Delta g,\ S_{\mathcal{F}}\sigma(s,t)\Delta g\rangle &= \sum_{(u,v)\in U\times V}|\langle\sigma(s,t)\Delta g,\ \sigma(u,v)f\rangle|^2\\ &= \sum_{(u,v)\in U\times V}\left|\left\langle\sigma\left(\frac{s}{u},t\right)\Delta g,\sigma(1,v)f\right\rangle\right|^2\\ &= \sum_{(u,v)\in U\times V}\left|\int|x|^{-\frac{1}{2}}\hat{g}\left(\frac{sx}{u}\right)\mathrm{e}^{2\pi\mathrm{i}tsxu^{-1}}\overline{\hat{f}(x)}\mathrm{e}^{-2\pi ivx}\,\mathrm{d}x\right|^2\\ &\le B_V\sum_{u\in U}\int\left|\hat{g}\left(\frac{sw}{u}\right)\right|^2\left|\hat{f}(w)\right|^2\frac{\mathrm{d}w}{|w|}.\end{aligned}$$

Similarly,

$$\langle\sigma(s,t)\Delta g,\ S_{\mathcal{F}}\sigma(s,t)\Delta g\rangle\ge A_V\sum_{u\in U}\int\left|\hat{g}\left(\frac{sw}{u}\right)\right|^2\left|\hat{f}(w)\right|^2\frac{\mathrm{d}w}{|w|}.$$

Choosing $\alpha_{s,t}$ so that

$$\alpha_{s,t}\ \langle\sigma(s,t)\Delta g,\ S_{\mathcal{F}}\sigma(s,t)\Delta g\rangle=\sum_{u\in U}\int\left|\hat{g}(sw)\right|^2\left|\hat{f}(uw)\right|^2\frac{\mathrm{d}w}{|w|}$$

we see $\alpha_{s,t}\in[B_V^{-1},A_V^{-1}]$. Choose $\lambda_{u,v}\in[F_T^{-1},E_T^{-1}]$ so that

$$\lambda_{u,v}\ \left\langle\sigma(u,v)\Delta f,\ S_{\mathcal{G}}\sigma(u,v)\Delta f\right\rangle=\sum_{s\in S}\int\left|\hat{g}(sw)\right|^2\left|\hat{f}(uw)\right|^2\frac{\mathrm{d}w}{|w|}.$$

Then

$$\begin{aligned}&\sum_{s\in S\cap a_MI_M}\alpha_{s,t}\ \langle\sigma(s,t)\Delta g,\ S_{\mathcal{F}}\sigma(s,t)\Delta g\rangle-\sum_{u\in U\cap a_MI_M}\lambda_{u,v}\ \left\langle\sigma(u,v)\Delta f,\ S_{\mathcal{G}}\sigma(u,v)\Delta f\right\rangle\\ &\quad=\sum_{s\in S\cap a_MI_M}\sum_{u\in U}\int\left|\hat{g}(sw)\right|^2\left|\hat{f}(uw)\right|^2\frac{\mathrm{d}w}{|w|}-\sum_{u\in U\cap a_MI_M}\sum_{s\in S}\int\left|\hat{g}(sw)\right|^2\left|\hat{f}(uw)\right|^2\frac{\mathrm{d}w}{|w|}.\end{aligned}$$

The technique used to prove Theorem 3.1 can be used to complete the proof. □

We can think of

$$p\text{-lim}\,\frac{1}{|U\cap a_MI_M|}\sum_{u\in U\cap a_MI_M}\lambda_{u,v}\ \left\langle\sigma(u,v)\Delta f,\ S_{\mathcal{G}}\sigma(u,v)\Delta f\right\rangle$$

as a value similar to $\mu_{\mathcal{F},\mathcal{G}}(p,c)$. With this understanding, Theorem 5.1 is analogous to Theorem 3.1.

Theorem 5.2. *Suppose the following conditions hold.*

(a) *U and S are relatively separated in $\mathbb{R}^+$.*
(b) *$f,g\in L^2(\mathbb{R})$ are admissible, form a localized pair, have compact support, and are Chui–Shi bounded with respect to U, S, respectively.*
(c) *$V\subset\mathbb{R}$ and $\mathcal{E}(V)=\left\{\mathrm{e}^{2\pi ivw}\right\}_{v\in V}$ is a frame for $L^2(\operatorname{supp}\hat{f})$ with bounds A_V, B_V.*
(d) *$T\subset\mathbb{R}$ and $\mathcal{E}(T)=\left\{\mathrm{e}^{2\pi itw}\right\}_{t\in T}$ is a frame for $L^2(\operatorname{supp}\hat{g})$ with frame bounds E_T, F_T.*

(e) $\mathcal{F} = \{\sigma(u,v)f\}_{(u,v)\in U\times V}$ *and* $\mathcal{G} = \{\sigma(s,t)g\}_{(s,t)\in S\times T}$ *are frames for for some common subspace of* $L^2(\mathbb{R})$ *with frame bounds* A, B *and* E, F*, respectively.*

Then for any free ultrafilter p *and any sequence* $a = \{a_M\} \subset \mathbb{R}^+$*, we have*

$$\frac{A\,E_T\,C_g}{F\,B_V\,C_f} \leq \frac{D_{\mathbb{R}^+}(U,a,p)}{D_{\mathbb{R}^+}(S,a,p)} \leq \frac{B\,F_T\,C_g}{E\,A_V\,C_f}.$$

Proof. Since $A \leq S_{\mathcal{F}} \leq B$ and $\alpha_{s,t} \in [B_V^{-1}, A_V^{-1}]$, we see

$$\alpha_{s,t}\ \langle \sigma(s,t)\Delta g,\ S_{\mathcal{F}}\sigma(s,t)\Delta g\rangle = \alpha_{s,t}\left\| S_{\mathcal{F}}^{\frac{1}{2}}\Delta g\right\|^2 \in \left[\frac{AC_g}{B_T}, \frac{BC_g}{A_T}\right].$$

We see

$$p\text{-lim}\,\frac{1}{2M}\left(\sum_{s\in S\cap a_M I_M}\alpha_{s,t}\ \langle \sigma(s,t)\Delta g,\ S_{\mathcal{F}}\sigma(s,t)\Delta g\rangle\right)$$
$$\in \left[\frac{AC_g}{B_T}, \frac{BC_g}{A_T}\right]\cdot D_{\mathbb{R}^+}(S,c,p).$$

Similarly,

$$\lambda_{u,v}\ \left\langle \sigma(u,v)f,\ S_{\mathcal{G}}\sigma(u,v)f\right\rangle = \lambda_{u,v}\left\| S_{\mathcal{G}}^{\frac{1}{2}}\Delta f\right\|^2 \in \left[\frac{EC_f}{F_T}, \frac{FC_f}{E_T}\right],$$

which implies

$$p\text{-lim}\,\frac{1}{2M}\left(\sum_{u\in U\cap a_M I_M}\lambda_{u,v}\ \left\langle \sigma(u,v)\Delta f,\ S_{\mathcal{G}}\sigma(u,v)\Delta f\right\rangle\right)$$
$$\in \left[\frac{EC_f}{F_T}, \frac{FC_f}{E_T}\right]\cdot D_{\mathbb{R}^+}(U,c,p).$$

Hence from Theorem 5.1, we obtain

$$\frac{A\,E_T\,C_g}{F\,B_V\,C_f} \leq \frac{D_{\mathbb{R}^+}(U,a,p)}{D_{\mathbb{R}^+}(S,a,p)} \leq \frac{B\,F_T\,C_g}{E\,A_V\,C_f}. \qquad \square$$

Corollary 5.3. *Suppose the hypotheses of Theorem 5.2 hold. Then for any free ultrafilter* p *and any sequence* $c = \{c_M\} \subset \mathbb{A}$ *we have*

$$\frac{A\,A_V\,E_T\,C_g}{F\,B_V\,F_T\,C_f} \leq \frac{D(U\times V,c,p)}{D(S\times T,c,p)} \leq \frac{B\,B_V\,F_T\,C_g}{E\,A_V E_T\,C_f}.$$

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